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NEW MEASURES OF NONCOMPACTNESS

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Abstract. In this paper are consider some new measures of noncompactness. They notations have proved useful in several areas of nonlinear functional analysis. Also, they measures of noncompactness suggested by some simple characterizations of relative compactness.

1. Introduction and main results

We notice, the first *measure of noncompactness*, in notation the function α, was defined and studied in 1930 by Kuratowski. Later in 1955 Darbo was the first who continued to use the function α.

Other measures were introduced by Goldenstein, Gohberg and Markus (the *ball measures of noncompactness, Hausdorff measure of noncompactness*; in notation χ) in 1957, Istrăţescu in 1957 and others; see Rakočević [10].

In this paper is the consider some new measures of noncompactness suggested by some simple characterizations of relative compactness.

Let (X, d) be a metric space and Q a bounded subset of X. The measure of noncomapctness of Q, denoted by $\tau(Q)$ or $T(Q)$, is the infimum of the set of all numbers $\varepsilon > 0$ such that Q can be covered by a finite number of sets with half diameters $< \varepsilon$, that is

$$\tau(Q)=\inf\Big\{\varepsilon > 0 : Q\subset \bigcup_{i=1}^{n} S_i,\ S_i \subset X, \mathrm{hdiam}(S_i) < \varepsilon \quad (i=1,\dots,n; n\in\mathbb{N})\Big\},$$

where $\mathrm{hdiam}(S_i) := \sup_{x\in S_i} d(x,y)$ for $y \in S_i$. Clearly we have the following inequalities which are connected with former results

$$\chi(Q) \le \tau(Q) \le \alpha(Q) \le \ \mathrm{diam}\ (Q)$$

for each bounded subset Q of X. Let Q_1 and Q_2 be two bounded sets in a complete metric space (X, d). Then

$$\tau(Q_1 \cup Q_2) = \max\Big\{\tau(Q_1), \tau(Q_2)\Big\}$$

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and if Q_2 reduces to a point $x_\circ$ we have $\tau(Q_1 \cup \{x_\circ\}) = \tau(Q_1)$. On the other hand: $\tau(Q) = 0$ if and only if $\overline{Q}$ is compact, $\tau(Q) = \tau(\overline{Q})$, $Q_1 \subset Q_2$ implies $\tau(Q_1) \leq \tau(Q_2)$ and

$$\tau(Q_1 \cap Q_2) \leq \min\Big\{\tau(Q_1), \tau(Q_2)\Big\}.$$

It is clear that the number $\tau(Q)$ has almost all properties of the numbers $\alpha(Q)$ and $\chi(Q)$ of Kuratowski and Hausdorff. The next statement is an extension of the well-known Cantor intersection theorem.

Theorem 1. *Let (X,d) be a complete metric space. If (F_n) is a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_{n\to\infty} \tau(F_n) = = 0$, then the intersection $F_\infty = \cap_{n=1}^{\infty} F_n$ is a nonempty and compact subset of X.*

The proof of this statement is totally analogous with the corresponding proofs for Kuratowski's and Hausdorff's functions; and thus we omit it.

If X is a normed space, then the function τ has some additional properties connected with the linear structures of a normed space. If Q, Q_1 and Q_2 are bounded subsets of a normed space X, then: $\tau(Q_1 + Q_2) \leq \tau(Q_1) + \tau(Q_2)$, $\tau(Q + x) = \tau(Q)$ for each $x \in X$, $\tau(\lambda Q) = |\lambda|\tau(Q)$ for each λ of the field $\boldsymbol{F}$ and $\tau(Q) = \tau(\text{conv } Q)$.

On the other hand, let (X,d) be a metric space. A mapping $f : X \to X$ is said to be **τ-contraction** if f is bounded and continuous, and there exists $q \in (0,1)$, such that for all bounded subset Q of X the following inequality holds

$$\tau\left(f(Q)\right) \leq q\tau(Q). \tag{1}$$

In connection with this, a mapping $f : X \to X$ is said to be **τ-condensing** if f is bounded and continuous and for every bounded subset Q of X for which $\tau(Q) > 0$:

$$\tau\left(f(Q)\right) < \tau(Q). \tag{2}$$

If X is a complex Banach space, C a nonempty bounded closed and convex subset of X and suppose that $f : X \to X$ is τ-condensing, then f has a fixed point and

$$\tau\left(\{x \in C : f(x) = x\}\right) = 0.$$

The proof of this statement can be modelled on the proof of the corresponding theorem for Kuratowski's function (number).

2. Still two measures of noncompactness

In further, let (X, d) be a complete metric space and Q a bounded subset of X. Let us recall that set Q is called **half diameter ε-discrete** (in notation $hd\varepsilon$-discrete), if

$$\sup_{x\in Q} d(x,y) \geq \varepsilon \quad \text{for } y \in Q,$$

and let us recall that set Q is called **diameter ε-discrete** (in notation $d\varepsilon$--discrete), if the following fact holds that diam $(Q) \geq \varepsilon$.

The measure of noncompactness of Q, denoted by $\tau hd(Q)$ or $Thd(Q)$, is defined in the following sense by

$$\tau hd(Q) = \inf\Big\{\varepsilon > 0 : Q \text{ has no infinite } hd\varepsilon\text{-discrete subsets}\Big\}.$$

On the other hand, the measure of noncompactness of Q, denoted by $\tau d(Q)$ or $Td(Q)$, is defined in the following sense by

$$\tau d(Q) = \inf\Big\{\varepsilon > 0 : \ Q \text{ has no infinite } d\varepsilon\text{-discrete subsets}\Big\}.$$

It is clear that functions $\tau hd(Q)$ and $\tau d(Q)$ have almost all properties of Kuratowski's and Istrăţescu's functions.

An open problem. *Compute $\tau hd(B_X)$ and $\tau d(B_X)$, where B_X is the closed unit ball in a normed space X of infinite dimension!*

We notice, if (X, d) is a metric space and Q is a bounded subset of X, then we obtain the following inequalities:

$$\beta(Q) \leq \tau hd(Q) \leq \tau d(Q),$$

where the function $\beta(Q)$ is called Istrăţescu's measure of noncomactness.

In conection with the preceding maps class of (1) and (2) we can introduce similar classes of mappings for functions $\tau hd(Q)$ and $\tau d(Q)$; and whenever the proof is not as for the preceding functions we mention this. Then we obtain some corresponding statements of fixed point for class' (1) and (2) of mappings.

The next statement, also, is an extension of the well-known Cantor intersection theorem.

Theorem 2. *Let (X, d) be a complete metric space. If (F_n) is a decreasing sequence of nonempty, closed and bounded subsets of X and if the following equality holds $\lim_{n\to\infty} \tau hd(F_n) = 0$ or $\lim_{n\to\infty} \tau d(F_n) = 0$, then the intersection $F_\infty = \cap_{n=1}^{\infty} F_n$ is a nonempty and compact subset of X.*

The proof of this statement can be modelled on the proof of the corresponding theorem for Kuratowski's functions; and thus we omit it.

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